Effect of Fluctuating Internal Coordinates on the Rotational Isomeric States Approximation

Marc L. Mansfield†

Department of Chemistry, Colorado State University, Fort Collins, Colorado 80523.
Received November 17, 1982

ABSTRACT: Methods for introducing independent fluctuations about the bond angles and bond rotational angles of rotational isomeric states models are discussed. Independent Gaussian fluctuations require only trivial adjustments of the rotation matrix. An examination of the Fourier components of the average bond projections and the eigenvalue spectrum of the $\mathbf{S} = \lambda^{-1}(\mathbf{U} \otimes \mathbf{E}_3)\|\mathbf{T}\|$ matrix demonstrates that chain models favoring helices will be particularly sensitive to perturbations, including fluctuations in both bond and bond rotational angles, changes in statistical weights, and errors induced by computing average rotation matrices with a discrete sum. Examples of models with a high sensitivity to such perturbations are given. These examples indicate that as a rule, the greater the tendency for a chain to favor a helix (or specifically, to occur in repeated rotational states), the less likely it will be that the chain can be accurately modeled by the standard rotational isomeric states approximation. They also indicate that models of high persistence length polymers should be viewed with at least some suspicion, due to the high sensitivity of such models to perturbations.

I. Introduction

A number of authors have been concerned with the effect of fluctuations in bond rotational angles upon the predictions of the rotational isomeric states (RIS) model.[1] On the one hand, Allegra and co-workers,[2-5] Bruckner,[6] and Oyama and Shiokawa[7] have considered more general representations (e.g., Fourier) of the transfer matrix, while on the other, Baram and Gelbart[8] and Freire and Fixman[9] have obtained results with fluctuating rotational angles using a rotation operator formalism. Tanaka[10] has developed a formalism permitting interdependent fluctuations on neighboring rotational angles. It is possible to calculate the effect of fluctuations in bond rotational angles upon a single given conformation, at least to second order in the fluctuation, and it was found that the end-to-end distance of such a conformation can be highly variable when fluctuations are permitted.[11] However, calculations[2,3,7,9-11] of the effect of fluctuations upon the root-mean-square end-to-end distance of an equilibrium ensemble of conformations indicated that the effect was much weaker, and it was concluded, in agreement with the original statement by Flory,[12] that the large deviations in the shape of any conformation cancel when an ensemble average over all conformations is performed.[11] For this reason and because the uncertainty in rotational potentials may not justify more detailed treatments, the RIS approach remains the method of choice for such calculations.

Nevertheless, fluctuations in bond angles and bond rotational angles have a much stronger effect upon the predicted dimensions of rigid chains (i.e., those chains with strong tendencies to form a helix). This has been demonstrated in at least two specific examples (DNA[13,14] and poly(L-proline)[15-18]) and in complete generality by Birshtein.[19] The reason for this is obvious: The quantity $\langle r^2 \rangle^{1/2}$ for a rigid helix is proportional to n; however, by permitting random fluctuations, no matter how slight, the helix eventually becomes a random coil with $\langle r^2 \rangle^{1/2}$ proportional to $n^{1/2}$. By the term helix, we refer to any chain with a repeated pattern of rotational states along the backbone, including therefore, degenerate cases such as the all-trans chain or the alternating ...g^+g^-... chain. Any chain with strong tendencies to form helices cannot be well represented by models that neglect fluctuations in rotational angles, and in some cases fluctuations in bond angles may also be important. This is one example of a general property of chain models with rigidity, namely, that such models are very sensitive to changes in the parameters of the model. In this paper we examine the origins of this sensitivity.

Two approximate methods have also been employed to treat the effect of fluctuations in rotational angles. In the first of these,[20-29] the value of the rotational angle averaged over the potential well is used instead of the value at the minimum, and the statistical weights include contributions from throughout the well. This "preaveraging" technique works well in some cases. One purpose of this paper is to identify those cases where it works best. In the second method, one computes the average transformation matrix directly, usually by summing over rotational angles at discrete intervals. A related approach is to perform the standard RIS matrix calculations with rotational states defined not only in the vicinity of the potential minima, but rather densely throughout conformational space. Of course, such calculations are exact in the limit of small intervals, but we show that intervals smaller than those often encountered may need to be used to accurately calculate the characteristic ratio. This is another example of the sensitivity of rigid-chain models to changes in the model.

In the next section of this paper we discuss a method for computing second moments of RIS chain molecules with independent fluctuations in both bond angles and bond rotational angles. This method provides assessment of the effect of fluctuating internal coordinates without scarificing the tractability of the RIS model, only requiring, for example, trivial adjustments of certain matrix elements.

In the third section we demonstrate the sensitivity of chain models with helix tendencies to changes in the model. We examine the Fourier components of the discrete function $P_j = b^{-2}\langle \mathbf{b}_i \cdot \mathbf{b}_{i+j} \rangle$ (for $\mathbf{b}_i$ and $\mathbf{b}_{i+j}$ bond vectors and b their length). These Fourier components can be expressed in terms of the eigenvalues of the matrix $\mathbf{S} = \lambda^{-1}(\mathbf{U} \otimes \mathbf{E}_3)\|\mathbf{T}\|$ (λ is the dominant eigenvalue of $\mathbf{U}$). Whenever any of these eigenvalues approaches unity, the chain favors a helix. Then Fourier components of P_j and the characteristic ratio become sensitive to perturbations, including, of course, perturbations due to fluctuating internal coordinates.

The fourth section presents examples of all the major points made in the paper. In some cases we show that models found in the literature are quite sensitive to fluctuations in rotational angles, and in other cases we examine

† Present address: Engineering Materials Program, Department of Chemical and Nuclear Engineering, University of Maryland, College Park, MD 20742.

0024-9297/83/2216-1863$01.50/0

fictitious chains that exhibit the properties under consideration.

Although this paper supports Birshtein's result[19] that rigid chains will be most affected by fluctuations, there seems to be no way of assessing the sensitivity of a model to fluctuations without actually performing the calculation. These calculations suggest that a model need not be too rigid before fluctuating rotational angles become important. (See the poly(oxymethylene) calculation in section IV.) Although the assumption of independent fluctuations is probably not accurate in most cases, the formalism discussed in section II will be useful in determining the sensitivity of a particular chain model to fluctuating internal coordinates. Since the formalism can be immediately adapted to any RIS model calculation, we would recommend that it be used as a matter of course in such calculations. If it indicates an unacceptable sensitivity to fluctuations, then one of the more sophisticated techniques mentioned above may be necessary.

II. Method of Calculation

The technique presented here is based on a hierarchy of averages. We average first over the fluctuations in any one particular chain conformation and then average over all chain conformations using standard RIS model formulas. The calculation of the square end-to-end vector, the square dipole moment, and the square radius of gyration for a chain in a particular rotational isomeric state involves serial products of transformation matrices given by eq I.25 of ref 1. Under the assumption of independent fluctuations, the fluctuation average of such a serial product is simply the product of the average matrices. It then follows that the fluctuation average is performed by replacing each $F(x)$ term appearing in the transformation matrix by $\langle F(x)\rangle_f$, where F represents either of the two functions sin or cos, and x represents θ_i or ϕ_i, where $\langle ...\rangle_f$ denotes an average over fluctuations. Then the average over all rotational isomeric states is performed in the usual fashion, with the transformation matrices modified as outlined above. Of course, in the case of Gaussian fluctuations, this corresponds to replacing each $F(x)$ term in the transformation matrix by $F(x)\exp(-X^2/2)$, where X represents the root-mean-square fluctuation in x. The fluctuations need not have the same distribution; they may vary from rotational state to rotational state about a given bond and from bond to bond. The only requirement is that they be independent. This technique applies only to quantities that are calculated from serial products of transformation matrices; higher moments, for example, will require special treatment.[9]

When the fluctuations are assumed to be independent, the statistical weights for first-order interactions should be assigned the value of the partition function $\int_{well} e^{-E/kT}\, d\phi$ and not simply $e^{-E_0/kT}$, where E_0 represents the energy at the conformational minimum. In the case of Gaussian fluctuations, this integral becomes $\Theta\Phi e^{-E_0/kT}$ (ignoring a common factor 2π), where Θ and Φ represent the root-mean-square fluctuation in either θ_i or ϕ_i. In cases where either Θ or Φ are the same for every rotational state of a particular bond, they may, or course, be omitted. Since we assume the fluctuations to be independent, the second-order statistical weights are unaffected.

Baram and Gelbart[8] and Freire and Fixman[9] have previously employed the above technique, but since their calculations employed rotational operators rather than the transfer matrices most often employed in RIS model calculations, it appears that the majority of workers using the transfer matrix formulas are unaware of the ease with which independent fluctuations may be introduced.

It is also possible,[30] under the assumption of independent, even fluctuations, to calculate the terms D_ϕ and D_θ in the series

$$\langle r^2\rangle = \langle r^2\rangle_{RIS} + D_\phi\Phi^2 + D_\theta\Theta^2$$

where $\langle r^2\rangle_{RIS}$ and $\langle r^2\rangle$ represent the mean square end-to-end distance of the chain model in the absence and presence, respectively, of fluctuations. (For even fluctuations there are no first-order terms.) The quantitites D_ϕ and D_θ may be computed as serial products of generator matrices, very similar to the way in which other RIS averages are performed.[1]

III. Sensitivity of Rotational Isomeric States Models to Perturbations

In this section, we consider the bond correlations of infinite chains in which all bonds are identical

$$P_j = b^{-2}\langle \mathbf{b}_i\cdot\mathbf{b}_{i+j}\rangle \tag{1}$$

and their Fourier components

$$F(k) = \sum_{j=-\infty}^{+\infty} P_j e^{-ijk} \tag{2}$$

for $\mathbf{b}_i$ and $\mathbf{b}_{i+j}$ bond vectors and b their length. Note that $P_0 = 1$ and $P_{-j} = P_j$. The inverse Fourier transform is given by

$$P_j = (2\pi)^{-1}\int_{-\pi}^{\pi} F(k)e^{ijk}\, dk \tag{3}$$

$F(k)$ is periodic on the interval 2π, is even and real, and obeys

$$2\pi = \int_{-\pi}^{\pi} F(k)\, dk \tag{4}$$

and

$$F(k) = 1 + 2\sum_{j=1}^{\infty} P_j \cos(jk) \tag{5}$$

Note that

$$F(0) = C_\infty \tag{6}$$

the characteristic ratio, and that $F(k)$ may be written as $\langle \mathbf{Q}^*\cdot\mathbf{Q}\rangle$, where $\mathbf{Q} = \sum_j e^{-ijk}\mathbf{b}_j$, from which it follows that

$$F(k) \geq 0, \qquad \text{for all } k \tag{7}$$

The quantity P_j is simply the 1,1 element (in Flory's[1] coordinate system) of the average of a series of transformation matrices, so it follows that $F(k)$ may be computed simply by multiplying each matrix $\mathbf{T}_j$, or equivalently, the matrix $\|\mathbf{T}\|$, by the scalar e^{ik}, and finally taking the real part of the result. Therefore[1]

$$F(k) = \mathcal{R}[(\mathbf{B}\otimes\mathbf{E}_3)(\mathbf{E}_{3\nu} + e^{ik}\mathbf{S})^{-1}(\mathbf{E}_3 - e^{ik}\mathbf{S})^{-1}(\mathbf{A}\otimes\mathbf{E}_3)]_{11} \tag{8}$$

where

$$\mathbf{S} = \lambda^{-1}(\mathbf{U}\otimes\mathbf{E}_3)\|\mathbf{T}\| \tag{9}$$

In the above, $\mathbf{E}_3$ and $\mathbf{E}_{3\nu}$ represent unit matrices of order 3 and 3ν, respectively, for ν the number of rotational isomeric states, $\|\mathbf{T}\|$ is the block diagonal matrix of which the jth block is the 3-space rotational transformation associated with the jth rotational state, $\mathbf{U}$ is the statistical weight matrix, λ is the dominant eigenvalue of $\mathbf{U}$, and $\mathbf{A}$ and $\mathbf{B}$ are the covariant and contravariant forms of the eigenvector of $\mathbf{U}$ associated with λ. The symbol $\mathcal{R}$ indicates that the real part of what follows should be taken.

Equation 8 may, of course, be written

$$F(k) = \mathcal{R}\sum_{j=1}^{3\nu} K_j(1 + e^{ik}s_j)(1 - e^{ik}s_j)^{-1} \tag{10}$$

where

$$K_j = \sum_{l,n=1}^{3\nu} (\mathbf{B} \otimes \mathbf{E}_3)_{1l} \mathbf{Z}_{lj} (\mathbf{Z}^{-1})_{jn} (\mathbf{A} \otimes \mathbf{E}_3)_{n1} \quad (11)$$

s_j are the eigenvalues of $\mathbf{S}$, and $\mathbf{Z}$ is the similarity transformation that diagonalizes $\mathbf{S}$. Note also that

$$P_l = \sum_{j=1}^{3\nu} K_j s_j^{|l|} \quad (12)$$

since $\langle \mathbf{T}^k \rangle = (\mathbf{B} \otimes \mathbf{E}_3)\mathbf{S}^k(\mathbf{A} \otimes \mathbf{E}_3)$,[31] and that $1 = \sum_j K_j$. The s_j are either real or occur in complex conjugate pairs. Each K_j is real if s_j is, and if s_i and s_j are complex conjugates, then so are K_i and K_j. Suppose there are n_r real and n_c complex eigenvalues. Let α_j denote the n_r real eigenvalues and let $\beta_j e^{\pm i\rho_j}$ denote the n_c complex eigenvalues. (Then α_j, β_j, and ρ_j are all real, β_j is positive, and $0 < \rho_j < \pi$.) Let K_{0j} and $K_{1j} \pm iK_{2j}$ be the K terms (eq 11) corresponding to α_j and $\beta_j e^{\pm i\rho_j}$, respectively. It follows that we may write

$$F(k) = \sum_{j=1}^{n_r} K_{0j} g(\alpha_j,k) + 2\sum_{j=1}^{n_c/2} [K_{1j}(g(\beta_j,k + \rho_j) + g(\beta_j,k - \rho_j)) + K_{2j}(h(\beta_j,k - \rho_j) - h(\beta_j,k + \rho_j))] \quad (13)$$

where

$$g(\alpha,k) = (1 - \alpha^2)/(1 - 2\alpha \cos k + \alpha^2) \quad (14)$$

and

$$h(\beta,k) = (\beta \sin k)/(1 - 2\beta \cos k + \beta^2) \quad (15)$$

For α a positive constant, g has a maximum at $k = 0$ and minima at $k = \pm\pi$ (or a minimum at $k = 0$ and maxima at $k = \pm\pi$ for α negative), with $g(\alpha,0) = (1 + \alpha)/(1 - \alpha)$ and $g(\alpha,\pm\pi) = (1 - \alpha)/(1 + \alpha)$. The function g is even in k, its integral is 2π on the interval $-\pi \le k \le \pi$, and $\lim_{\alpha\to 1} g(\alpha,k) = 2\pi\delta(k)$, so that $g(\alpha,k)$ is sharply peaked at $k = 0$ when α is near 1. The function $h(\beta,k)$ is odd in k, and so, of course, has zero integral on the interval $-\pi \le k \le \pi$. It has zeros at $k = 0$ and $k = \pm\pi$ when $\beta \neq 1$. When $\beta = 1$, the zeros at $\pm\pi$ are still present, but it behaves as k^{-1} for k near zero. It has a maximum at $k_{max} = \cos^{-1}(2\beta/(1 + \beta^2))$ (and a corresponding minimum at $-k_{max}$). Note that $k_{max} \to 0$ and $h(\beta,k_{max}) \to \infty$ as $\beta \to 1$. It follows that for β near 1, $h(\beta,k)$ has a peak at $k \cong 1 - \beta$ (and a corresponding trough at $k \cong \beta - 1$) of approximate height $[2(1 - \beta)]^{-1}$.

Consider now the application of the above arguments to a helical chain, or more precisely, a chain that repeats perfectly the rotational state of each bond. For such a chain, $\mathbf{S}$ degenerates to the rotational transformation $\mathbf{T}$, and if we let s_i and $\mathbf{v}_i$ represent the eigenvalues and eigenvectors of $\mathbf{S} = \mathbf{T}$, all the following holds: $s_1 = 1$, $s_2 = e^{i\rho}$, $s_3 = e^{-i\rho}$; $\mathbf{v}_1$ is the rotation axis of $\mathbf{T}$; $\mathbf{v}_2$ and $\mathbf{v}_3$ span the plane perpendicular to $\mathbf{v}_1$; $\mathbf{v}_2 = \mathbf{v}_3^*$; ρ is the angle of rotation of transformation $\mathbf{T}$; and the diagonalizing similarity transformation $\mathbf{Z}$ is unitary.[32] For this case, eq 11 becomes

$$K_j = |(\mathbf{v}_j)_1|^2 \quad (16)$$

so that b^2K_1 represents the square projection of the bond vector onto the rotation axis and $b^2K_2 = b^2K_3$ is one-half the square projection of the bond vector onto the plane perpendicular to the rotation axis. Specializing eq 13 to this case yields

$$F(k) = K_1 g(1,k) + 2[K_2 g(1,k + \rho) + g(1,k - \rho)] = 2\pi[K_1\delta(k) + 2K_2\delta(k + \rho) + 2K_2\delta(k - \rho)] \quad (17)$$

This equation may also be derived from the fact that for a particular coordinate system (z axis aligned along the rotation axis of $\mathbf{T}$, and $\mathbf{b}_0$ in the x–z plane) the jth bond vector has coordinates ($br \cos(j\rho)$, $br \sin(j\rho)$, bz) so that $P_j = r^2 \cos(j\rho) + z^2$. The Dirac δ functions result upon Fourier transformation, and we have already established that $K_1 = z^2$ and $2K_2 = r^2$.

From this we expect three sharply peaked g functions centered at $k = 0$ and $\pm\rho$, respectively, to be the dominant features of any RIS model favoring a helix characterized by the angle ρ. $F(k)$ in the vicinity of $k = 0$ and $k = \pm\rho$ will be given almost entirely by these three functions. Now consider introducing fluctuating rotational angles in such a model. For small fluctuations, this represents a perturbation of strength proportional to $-\Phi^2$ applied to the matrix $\mathbf{S}$. To first order in the perturbation, each of the terms K_{0j}, K_{1j}, K_{2j}, α_j, β_j, and ρ_j in eq 13 will be modified by a factor $1 + c\Phi^2$, for some c. In particular, consider the effect of this perturbation on $g(\alpha,k)$. Let α and $\alpha - a\Phi^2$ represent the unperturbed and the perturbed eigenvalues, respectively. If the unperturbed chain is a helix, then $\alpha = 1$, and g is given, to first order in Φ^2, by

$$g(1 - a\Phi^2,k) = 2/(a\Phi^2), \quad \text{for } k \text{ near zero}$$
$$g(1 - a\Phi^2,k) = a\Phi^2/(1 - \cos k), \quad \text{for } k \text{ not near zero} \quad (18)$$

i.e., g behaves as $\Phi^{\pm 2}$. If the unperturbed chain approximates a helix, then $\alpha = 1 - \epsilon$, for ϵ small, and one obtains

$$g(1 - \epsilon - a\Phi^2,k) = (2/\epsilon)(1 - a\Phi^2/\epsilon), \quad \text{for } k \text{ near zero}$$
$$g(1 - \epsilon - a\Phi^2,k) = \epsilon(1 - \cos k)^{-1}(1 + a\Phi^2/\epsilon), \quad \text{for } k \text{ not near zero} \quad (19)$$

and the perturbation is amplified by a large factor ϵ^{-1}. Even though eq 19 indicates an amplified perturbation at essentially all values of k, this amplification may be partially or totally masked at values of k not near 0 or $\pm\rho$ by other terms appearing in eq 13. Therefore, $F(k)$ for chains favoring a helix characterized by the angle ρ is most sensitive to fluctuating rotational angles at $k = 0$ (the characteristic ratio) and at $k = \pm\rho$. We also observe that $h(1 - \epsilon - a\Phi^2,k_{max})$ exhibits the same ϵ^{-1} amplification. However, since $F(k)$ is nonnegative, h functions should not make large contributions to eq 13, so that this amplification of h is probably not significant.

The results of this section should apply to other sorts of perturbations (e.g., changes in statistical weights or inclusion of additional rotational states) as well. The amplification of a perturbation as obtained in eq 19 should be quite general, implying that RIS models tending to a helix will be very sensitive to any changes in the model. Specific examples of this are calculations in which characteristic ratios of either poly(L-proline) or the poly(n-alkyl isocyanates) are very sensitive to either a small number of additional rotational states, helix reversals, or changes in potential energy functions.[15-17,33,34] The perturbation may also be the result of an approximate integration. Small integration errors in the matrix $\mathbf{S}$ (or its analogue $\langle \mathbf{T} \rangle$ in polymers with separable rotational potentials) can be amplified substantially as an example in the next section demonstrates.

Although this section applies only to chains in which all bonds are the same, the generalization of this argument to chains with regular repeat units longer than a single bond is not difficult, since equations analogous to eq 8 are available for such chains.[35] As for chains without regular repeat units, the rule should also apply to block copolymers in which one or more blocks tend to be helical, while random copolymers are not expected to strongly favor a helix.

Some comment on the expected magnitude of Φ and Θ, and therefore the expected perturbation strength induced by fluctuations, may be in order. Threefold rotational barriers of height $2kT$, $3kT$, and $5kT$ give $\Phi = 19.1°$, 15.6°, and 12.1°, respectively (assuming only the quadratic part of the potential to be important). In the examples that follow we usually take 15° as a representative value, but this is only used for purposes of demonstration. Using literature values[36] for bond bending potentials, one obtains $\Theta = 4°$ as a representative value. These correspond to perturbations of strength $(15\pi/180)^2 = 6.9\%$ and $(4\pi/180)^2 = 0.5\%$, respectively. Calculations[2,3,6,9-11] of the effect of fluctuating rotational angles on flexible chains indicate that the error in the characteristic ratio is usually less than 10%, while the effect of fluctuating bond angles is negligible. The estimated perturbation strengths above are certainly consistent with this. However, fluctuating rotational angles may become important with only rather weak amplification of the perturbation (a/ϵ around 3), i.e., with only rather weak helical character. Fluctuating bond angles will become important when amplifications greater than about $a/\epsilon = 30$ are encountered.

IV. Examples

The following examples demonstrate the effects discussed above. Some of these are taken from the literature, but we do not suggest that any of these models, either with or without fluctuations, are good or bad models of the molecules they represent. We only point out that a class of RIS models exists whose members are sensitive to fluctuations in internal coordinates. We follow the notation employed by the authors of each model, defining explicitly any deviations or additions in notation. We employ the convention in which rotational angles at trans are zero and in which θ_i represents the supplement of a bond angle.

A. Symmetric Threefold Rotational Potentials. The conformational states of chains for which all the bonds are identical and for which the potentials are threefold and symmetric are controlled by three statistical weights, σ, ω, and ψ,[37] as well as the geometrical parameters θ and ϕ_i, for $i = 1$, 2, or 3. Employing the polyethylene parameters given by Flory,[1] $\sigma = 0.54$, $\omega = 0.088$, $\psi = 1$, $\theta = 68°$, and $\phi_i = 0°$ or $\pm 120°$, and extrapolating to $n = \infty$, we obtain $C_\infty = 6.89 - 16.8\Theta^2 - 6.02\Phi^2$. Therefore, for these particular statistical weights, the characteristic ratio is in error by about 6% for $\Phi = 15°$. This is probably comparable to experimental error for this polymer. We expect $\Theta^2 \ll \Phi^2$ so that neglect of fluctuations in bond angles should give small errors. For example, $\Theta = 4°$ gives an error of 1.2%. Indeed, for such values of σ, ω, and ψ, fluctuating rotational angles have little effect. However, calculations indicate that the error is great in three different regions of the σ-ω-ψ space. These regions are first, $\sigma \cong 0$, irrespective of the values of ω and ψ, second, when $\sigma \gg 1$, $\psi \gg 1$, and $\omega \lesssim 1$, and third, when $\sigma \gg 1$, $\omega \gg 1$, and $\psi \lesssim 1$. This behavior is demonstrated in Figures 1 and 2. The three cases correspond to a strong preference for ...tt..., ...g$^\pm$g$^\pm$..., and ...g$^\pm$g$^\mp$... helices, respectively. This last helix has zero pitch (any six consecutive bonds assume the cyclohexane chair conformation) so that the correction is positive. Note that large corrections are observed even for chains with rather modest characteristic ratios. (E.g., the g$^\pm$g$^\pm$ chain has $C_{100} = 12$, and $\Phi = 15°$ applies a -30% correction to the value of C_{100}.)

B. Poly(oxymethylene). For this example we consider the model discussed by Flory,[38] employing the notation given in ref 38. The gauche states were assigned to be $\pm 120°$, the bond angles were assigned to be 110° ($\theta_a = \theta_b = 70°$), and the statistical weight parameters σ and ω were assigned the values 12 and 0.05, respectively. Such statistical weights correspond to a chain in which gauche states are preferred much more strongly than trans and in which g$^\pm$g$^\pm$ pairs are much more strongly preferred than g$^\pm$g$^\mp$ pairs. This chain consists of helical sequences, each about seven bonds long on the average, of bonds in the same gauche state, separated either by a trans bond or by a g$^\pm$g$^\mp$ pair.[38] Figure 3 demonstrates the change in C_∞ if we modify this model by permitting independent fluctuations of equal strength about each bond rotational angle. For this model $\epsilon = 0.15941$ and $a/\epsilon = 3.3$. When $\Phi = 15°$, the error in C_∞ is about 19%.

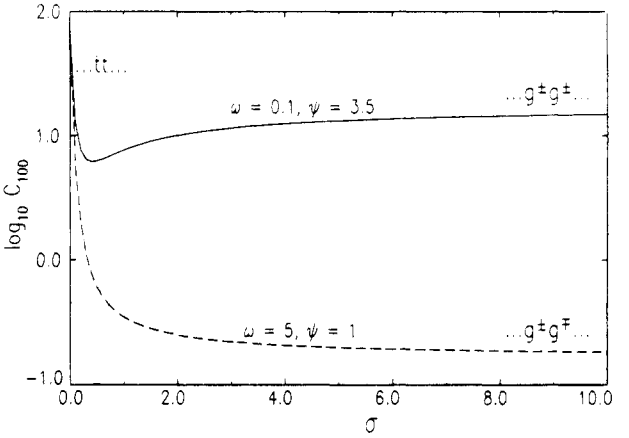

Figure 1. Characteristic ratios in the absence of rotational angle fluctuations of simple 100-bond chains with threefold, symmetric potentials, for various values of the statistical weights σ, ω, and ψ. For certain values of the statistical weights, given in the text, the chains can be made to favor one of the three helices shown.

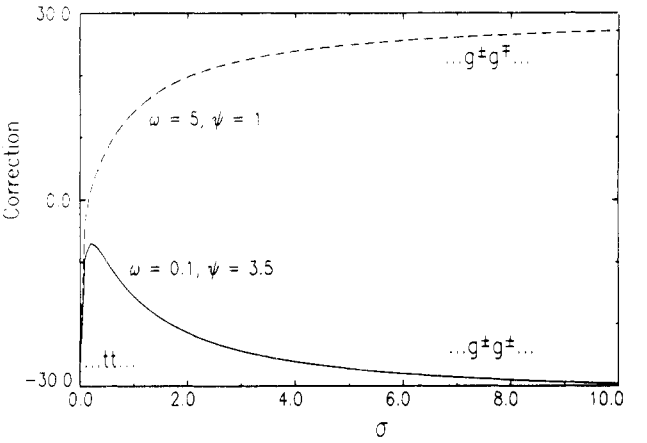

Figure 2. Percent correction to the characteristic ratios plotted in Figure 1 when independent, Gaussian fluctuations of root-mean-square magnitude 15° are applied to each bond.

Allegra et al.[3] have also computed C_∞ for a model of this polymer, using the Fourier representation[2,6] of the transfer matrix, thereby including contributions from all possible values of the rotational angles. They compared the results to a RIS calculation and found rather small differences between the predicted unperturbed dimensions. However, the gauche–trans energy difference in their model was less than kT, indicating that their model has much less helical character than the one referred to in the preceding paragraph.

The preceding two examples indicate that chains with only rather small tendencies to form helices (e.g., with characteristic ratios around 10) can be sensitive to fluctuating rotational angles.

C. Poly(tetrafluoroethylene). Bates and Stockmayer[39,40] have presented a four-state model of the PTFE chain with rotational states at $\pm 15°$ and $\pm 115°$. The statistical

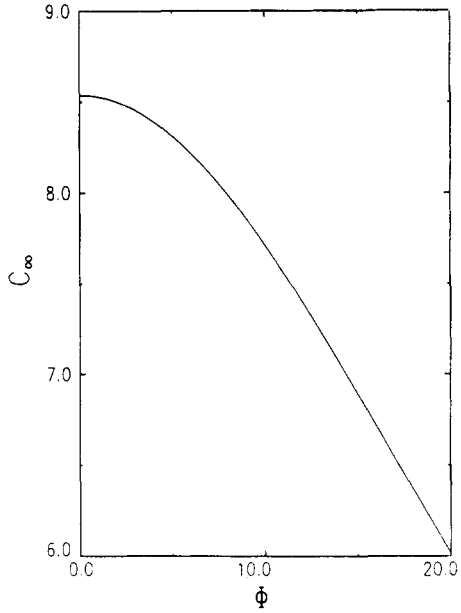

Figure 3. Change in the characteristic ratio of a model of poly(oxymethylene) when independent, Gaussian fluctuations of root-mean-square magnitude Φ (in degrees) are applied to each bond.

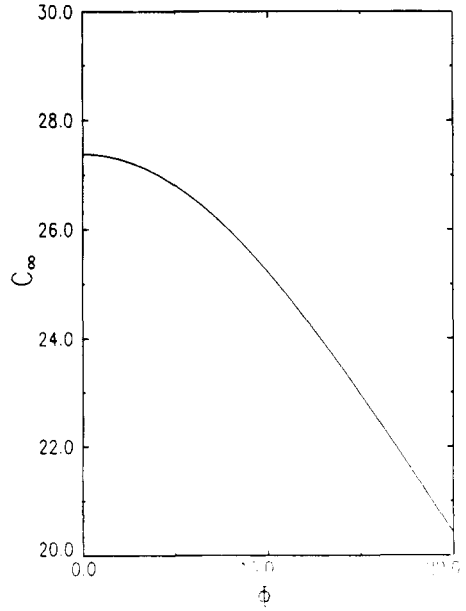

Figure 4. Change in the characteristic ratio of a model of poly(tetrafluoroethylene) when independent, Gaussian fluctuations of root-mean-square magnitude Φ (in degrees) are applied to each bond.

weights σ and ω are defined in such a way that the chain prefers the two trans states strongly. Using the values E_σ = 1.4 kcal mol^{-1}, E_ω = 1.1 kcal mol^{-1}, and T = 600 K and permitting independent Gaussian fluctuations in each of the bonds yield the data displayed in Figure 4. For this model, ϵ = 0.05043 and a/ϵ = 2.7. When Φ = 15°, the error in C_∞ is about 16%.

D. Poly(L-proline II). In the following example, we let ϕ, ψ, and ω represent the rotational angles about the N–C$^\alpha$, and C$^\alpha$–C′, and the C′–N bonds, respectively. To be consistent with the rest of the paper, we will depart from the usual convention for polypeptide nomenclature[41] and set the value of these three rotational angles equal to zero at the trans state.

The poly(L-proline) molecule exists in two conformations in both crystalline and dissolved states, depending upon

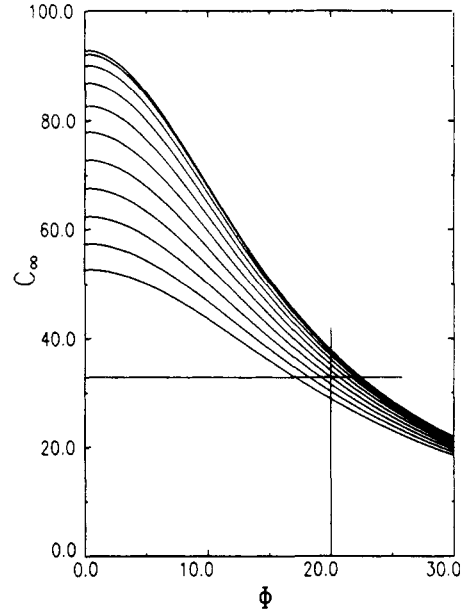

Figure 5. Change in the characteristic ratio of a model of poly(L-proline II) when independent, Gaussian fluctuations of root-mean-square magnitude Φ or Ω are applied to the ϕ or ω bonds, respectively. The values of Φ (in degrees) are displayed on the ordinate. Each curve, from top to bottom, corresponds to Ω = 0°, 1°, ..., 10°, respectively. The characteristic ratio corresponding to Φ = 20°, Ω = 7° is indicated.

whether the peptide bond is cis (form I) or trans (form II). The various theoretical predictions[15-17,42,43] of the characteristic ratio of this chain provide good examples of the role played by fluctuating rotational angles. In an early calculation[42] of the characteristic ratio of form II it was assumed that both the rotation angles ω and ϕ were rigid (the first because the peptide bond has partial double-bond character and the second because the N–C$^\alpha$ bond is part of a pyrrolidine ring), and conformational energies were computed for changes in the rotational state of the angle ψ. A single conformational well centered at about ψ = 310° was located, yielding a structure in good agreement with X-ray crystallographic studies. The computed characteristic ratio was about 100[42] (defined as $\langle R^2\rangle_0/n_p l_p^2$ for n_p the number of residues and l_p = 3.80 Å), much larger that the experimental value of ca. 14 in H_2O and ca. 18 in organic solvents.[15] It was shown that by including fluctuations in ω and ϕ, introducing a small number of cis residues, and permitting a small population of ψ in an additional conformational state near 130°, characteristic ratios in line with experiment could be obtained.[15-17,43] Experimental evidence has been given for the existence both of cis residues[44] and of the additional rotational state of ψ,[43] although the existence of cis residues has been disputed.[45] As an example of the sensitivity of such models to fluctuations in rotational angles, we consider the model of Schimmel and Flory[42] but include fluctuations in both ω and ϕ. The results are shown in Figure 5, in which Ω and Φ denote the root-mean-square fluctuation in ω and ϕ, respectively. If we assume Φ = 20° and Ω = 7° (consistent with calculations on the pyrrolidine ring[43] and the usual value given for the barrier to rotation about the amide bond[46]), the characteristic ratio drops from ca. 90 to ca. 30. (We are unable to reproduce exactly the Φ = Ω = 0° value given by Schimmel and Flory[42] because of errors in reading their potential energy curve.) For this model, ϵ = 0.01420 and a/ϵ = 12. A very substantial drop in the characteristic ratio is obtained without recourse either to the additional ψ conformation or to cis–trans isomerization, although it does appear that one or the other of the above

Table I
Least-Squares Fit of $C_\infty l^2$ and $D^{1/2}l$ for Models of Poly(n-alkyl isocyanates)[a]

ϕ_1, ϕ_2	θ_1, θ_2	$C_\infty l^2$, Å^2	$D^{1/2}l$, Å	Φ
40, 200	66, 66	785	329	4.88
40, 200	60, 74	783	335	4.78
95, 95	66, 66	793	295	5.59
95, 95	60, 74	793	299	5.42
40, 40	66, 66	799	260	7.74
40, 40	60, 74	798	268	7.11

[a] The first two columns give the geometry of the model, and the third and fourth give the best-fit quantities $C_\infty l^2$ and $D^{1/2}l$, respectively (experimental values are 747 Å^2 and 420 Å[48,49]). The last column gives the best fit root-mean-square fluctuation in ϕ_1 and ϕ_2. All angles are displayed in degrees. The rotational angles 40, 200; 95, 95; and 40, 40 are from the models of Troxell and Scheraga,[50] Han and Yu,[51] and Tonelli,[34] respectively. The bond angles 66, 66; and 60, 74 are from Troxell and Scheraga[50] and Han and Yu,[51] respectively. All bond lengths in the calculations were 1.32 Å.

is necessary to obtain good agreement. As a general rule any model with helix-forming tendencies as strong as that of Schimmel and Flory[42] will be very sensitive to fluctuations in rotational angles and to even a small amount of conformational diversity.

E. Poly(n-alkyl isocyanates). It is by now well established that poly(n-alkyl isocyanate) chains are stiff, and most likely helical, exhibiting a transition from rodlike to random coil behavior at molecular weights around 10^5.[33] It has been proposed that the flexibility at high molecular weights is due to reversals in helix sense and/or inclusion of a small number of repeat units in which the C=O bond, and not the C=N bond, of the isocyanate monomer has been inserted in the backbone.[33,34,47] As noted above, the average dimensions of an otherwise helical chain will be very sensitive to even a small number of such imperfections. It has been customary to characterize these chains in terms of the Kratky–Porod wormlike chain.[33,47,48] The mean dimensions of an unperturbed chain can be expressed as $\langle R^2\rangle_0 = nl^2C_\infty - 2Dl^2$, where the quantities C_∞ and D can be computed by the RIS model and are directly related to the two independent parameters of the wormlike chain model.[49] Using experimental wormlike chain parameters obtained from light scattering measurements[48] provides the values $C_\infty l^2 = 747$ Å^2 and $D^{1/2}l = 420$ Å. Independent Gaussian fluctuations were applied to each rotational angle of three different proposed helices[34,50,51] with the magnitude of the root-mean-square fluctuation adjusted to give the best least-squares fit to the two quantities $C_\infty l^2$ and $D^{1/2}l$. As may be seen in Table I, only modest fluctuations are needed to give moderately good agreement with the experimental quantities, without recourse either to helix reversals or to monomer units with different structures. It follows that such fluctuations play a major role in determining the chain flexibility at large molecular weights. This calculation does not, of course, rule out the existence of such helix imperfections, but it does indicate that any complete picture of the flexibility of this chain at large molecular weights must include fluctuations in rotational angles.

F. Effect of "Preaveraging" Fluctuations. As noted in the Introduction, one technique used to account for fluctuations employs the value of the rotational angle averaged over the conformational well, rather than the value of the angle at the minimum. Since the potentials controlling fluctuations in the above examples are even, the average rotational angle in each case is just the value at the minimum. It follows that "preaveraging" is unable

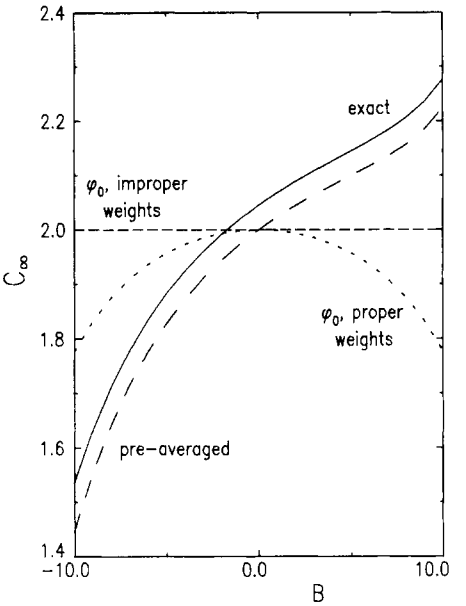

Figure 6. Exact and approximate characteristic ratios of a chain model, comparing the accuracy of "preaveraging" and other approximate techniques. See the text for an explanation of the labels on the curves.

to reproduce any of the results in the above examples, giving the $\Phi = 0$ result for the characteristic ratio at any value of ϕ. A rigid helix treated by "preaveraging" remains a rigid helix. This approach is sensitive only to the anharmonicity of the potential and should not be used when the chain is sensitive to harmonic fluctuations, i.e., for chains with high helical character. The following example demonstrates the performance of this technique applied to flexible chains with anharmonic potentials.

Consider a three-states chain with tetrahedral bond angles and separable rotational potentials in which the potential energy in the vicinity of the trans state is given (in units of kT) by

$$E_t(\phi) = (72/\pi^2)\phi^2$$

and in the vicinity of the gauche(±) states by

$$E_\pm(\phi) = (72/\pi^2)(\phi - \phi_0)^2 \pm B(\phi - \phi_0)^3/(1 + (\phi - \phi_0)^2)$$

for ϕ measured in radians and for $\phi_0 = \pm 2\pi/3$. The parameter B controls the anharmonicity of the two gauche potential wells. Note that the anharmonic term does not shift the position of the minimum. For all three wells, $\Phi = 15°$ when $B = 0$. The potential is zero at $\phi = 0°$ and at $\phi = \pm 120°$, so the chain is flexible and not strongly sensitive to perturbations. Averages over the wells were approximated by setting integration limits at $\pm\infty$. The characteristic ratios computed for this model by four alternate methods are displayed in Figure 6. The four methods are (1) defining rotational isomeric states at the potential minima and statistical weights as $e^{-E_0/kT}$ for E_0 the energy at each minimum, labeled "ϕ_0, improper weights" in Figure 6, (2) defining rotational isomeric states at the potential minima and statistical weights as $\int_{well} e^{-E/kT}$, labeled "ϕ_0, proper weights" in Figure 6, (3) defining rotational states in terms of averages over the well and statistical weights as $\int_{well} e^{-E/kT}$, which is the "preaveraging" technique and is so labeled in Figure 6, and (4) the exact method outlined in section II and so labeled in Figure 6. Of these alternatives, only the "preaveraging" technique does well. The error in the calculated characteristic ratio is due to the harmonic part of the potential. This calculation indicates that "preaveraging" gives reasonably accurate results for flexible chains.

Table II
Approximate and Exact Values of $C = \langle \cos \phi \rangle$ for the Trans State of a Model Polymer[a]

interval	C
10°	0.98611
5°	0.98799
3°	0.98839
1°	0.98859
exact	0.98862

[a] Degree values in the first column are the size of the interval used in the trapezoidal rule integration of C.

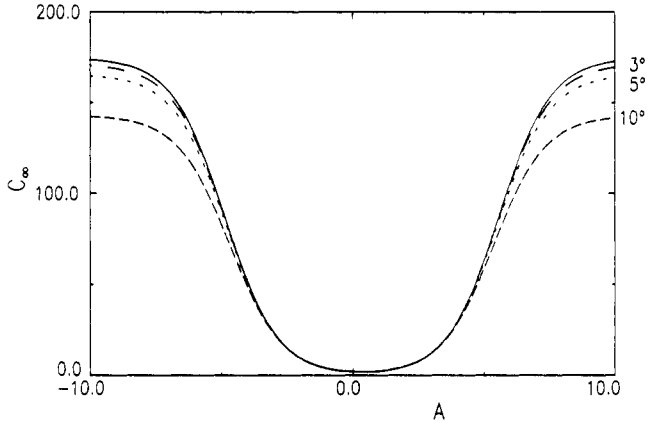

Figure 7. Characteristic ratios of a chain model demonstrating the errors that can result by computing average rotation matrices from discrete sums. The solid curve is the exact result, while the dashed curves are the results from three different trapezoidal rule integrations performed at the intervals indicated. The curve from the 1° integration (not shown) lies very close to the exact curve.

G. Errors Induced through Numerical Integration. Consider a polymer chain with separable rotational potentials, with tetrahedral bond angles, and with two admissible states, trans and cis. To each rotational state apply fluctuations such that variations up to ±15° are allowed at no cost of energy (square well potentials with infinite walls of total width 30° centered at $\phi = 0°$ or 180°). We assume that each tt or cc pair of bonds contribute nothing to the energy of the chain, while each tc or ct pair contributes AkT. Then for $A \ll 0$, the chain resembles a helix with alternating cis–trans states, while for $A \gg 0$, the chain contains long alternating cis and trans helices. Let $C = \langle \cos \phi \rangle_t = -\langle \cos \phi \rangle_c$ and note that $\langle \sin \phi \rangle_t = \langle \sin \phi \rangle_c = 0$, where $\langle \ldots \rangle_t$ and $\langle \ldots \rangle_c$ represent averages over the trans and cis wells, respectively. Now $C = (12/\pi) \sin (\pi/12)$, but in order to examine the effects of approximate integration, we also approximate C by trapezoidal rule integration, using 10°, 5°, 3°, and 1° intervals. Table II compares the approximate and exact values of C obtained, and Figure 7 gives the characteristic ratio as a function of A, for both exact and approximate values of C. Even though C computed at 10° intervals is wrong by only about one part in 400, the characteristic ratio for extreme values of A is wrong by about one part in 5. The integration error has been amplified approximately 80-fold by the effect discussed in section III. This amplification is observed only when the chain begins to resemble a helix, i.e., for extreme values of A. At least for this particular chain, integration at 10° intervals is inappropriate. This is significant because integrations at 10° are common in the literature.

H. Effect of Fluctuating Bond Angles. Consider the same chain model as in example G, but with harmonic fluctuations of root-mean-square strength Θ in the bond angles. Figure 8 gives C_∞ as a function of Θ for $A = -10$.

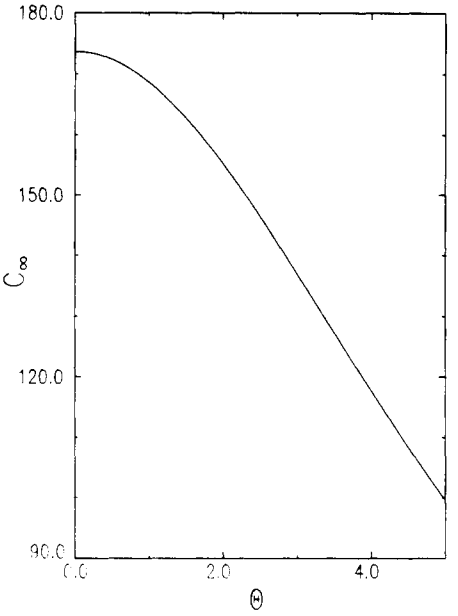

Figure 8. Change in the characteristic ratio of a model polymer when Gaussian fluctuations in the bond angles of root-mean-square strength Θ (in degrees) are applied.

For this model, $\epsilon = 0.00511$ and $a/\epsilon = 96$. At Θ = 4°, C_∞ has dropped by about $^1/_3$.

V. Conclusions

The artifice of replacing the continuum of rotational states of polymer chains by a small number of discrete states, employed in rotational isomeric state calculations, may not always give accurate results. In particular, chain models tending to favor a helical structure are sensitive to fluctuations in the internal coordinates, the sensitivity being correlated with the degree of helicity. This sensitivity extends to chain models that are random coils at usual molecular weights and that exhibit helical sections along the chain, but it is most pronounced in chain models that are rodlike. It has been recognized for some time that individual chain conformations exhibit large fluctuations in shape when even small fluctuations in rotational angles are introduced.[11,12] The justification of the RIS model lies in the lack of correlation between these large fluctuations when averaged over all chain conformations.[11,12] However, these large fluctuations are correlated to some degree in chain models with helical tendencies, so that RIS calculations become inaccurate.

A number of molecules are known to behave as rigid rods at low molecular weights and become random coils when the molecular weight is increased. Specific examples include double-helical DNA[13,52,53] and polyisocyanates.[33] Birshtein's results,[19] Olson's calculations[13,14] on DNA, and the calculations presented here demonstrate that rather small fluctuations in the rotational angles about an otherwise rigid helix should always be included in attempts to explain the flexibility at high molecular weights.

The characteristic ratios of chains with large persistence lengths are sensitive to even minor changes in the model. Changes in the energy of conformational states lying at high energies by, say, 1 kcal/mol, changes in the fluctuation strength permitted in internal coordinates, and even numerical integration errors can significantly alter the calculated characteristic ratio of large persistence length polymers. This does not imply that such models are not useful. They most likely retain at least qualitative accuracy. Nor does it necessarily call for more accurate numerical integrations. In light of the inaccuracies in estimating rotational potentials and the computational effort

required, perhaps 10° integration intervals are justified. But it probably does mean that model parameters obtained by fitting the experimental characteristic ratios of large persistence length chains should be regarded with at least some suspicion.

We have also shown that the "preaveraging technique" is inappropriate to large persistence length chains but that it can give accurate results if the chain is flexible. It does not account for the changes induced by the harmonic part of the rotational potential but these are small if the chain is flexible. Probably the most significant problem with this technique is that the errors it generates can only be determined by performing a more exact calculation.

It has been shown that long-range order does not exist in systems of one or two dimensions.[54-57] Thermal fluctuations in the relative positions of neighboring components of the system are amplified by the dimensionality so that the mean square displacement of any individual component grows without limit as the size of the system increases. The effect discussed here is a special case of this phenomenon.

Acknowledgment. The author thanks Prof. Marshall Fixman for reviewing the manuscript and for calling his attention to the work[54-57] on long-range order in one and two dimensions. This work was supported by NIH Grant No. GM27945.

Registry No. Poly(tetrafluoroethylene) (homopolymer), 9002-84-0; poly(L-proline) (homopolymer), 25191-13-3; poly(L-proline) (SRU), 25213-33-6.

References and Notes

(1) Flory, P. J. "Statistical Mechanics of Chain Molecules"; Interscience: New York, 1969.
(2) Allegra, G.; Immirzi, A. *Makromol. Chem.* **1969**, *124*, 70.
(3) Allegra, G.; Calligaris, M.; Randaccio, L. *Macromolecules* **1973**, *6*, 390.
(4) Allegra, G.; Calligaris, M.; Randaccio, L.; Moraglio, G. *Macromolecules* **1973**, *6*, 397.
(5) Allegra, G.; Bruckner, S. *Macromolecules* **1977**, *10*, 106.
(6) Bruckner, S. *Macromolecules* **1981**, *14*, 449.
(7) Oyama, T.; Shiokawa, K. *Polym. J.* **1981**, *13*, 1145.
(8) Baram, A.; Gelbart, W. M. *J. Chem. Phys.* **1977**, *66*, 4666.
(9) Freire, J.; Fixman, M. *J. Chem. Phys.* **1978**, *69*, 634.
(10) Tanaka, G., unpublished results.
(11) Mansfield, M. L. *J. Chem. Phys.* **1980**, *72*, 3923.
(12) Reference 1, p 57.
(13) Olson, W. K. In "Nucleic Acid Geometry and Dynamics"; Sarma, R. H., Ed.; Pergamon Press: New York, 1980.
(14) Olsen, W. K. *Biopolymers* **1979**, *18*, 1213.
(15) Mattice, W. L.; Mandelkern, L. *J. Am. Chem. Soc.* **1971**, *93*, 1769.
(16) Mattice, W. L.; Nishikawa, K.; Ooi, T. *Macromolecules* **1973**, *3*, 443.
(17) Tanaka, S.; Scheraga, H. A. *Macromolecules* **1975**, *8*, 623.
(18) Darsey, J. A.; Mattice, W. L. *Macromolecules* **1982**, *15*, 1626.
(19) Birshtein, T. M. *Polym. Sci. USSR (Engl. Transl.)* **1974**, *16*, 60.
(20) Suter, U. W.; Flory, P. J. *Macromolecules* **1975**, *8*, 765.
(21) Yoon, D. Y.; Sundararajan, P. R.; Flory, P. J. *Macromolecules* **1975**, *8*, 776.
(22) Yoon, D. Y.; Suter, U. W.; Sundararajan, P. R.; Flory, P. J. *Macromolecules* **1975**, *8*, 784.
(23) Sundararajan, P. R. *Macromolecules* **1977**, *10*, 623.
(24) Sundararajan, P. R. *Macromolecules* **1978**, *11*, 256.
(25) Ojalvo, E. A.; Saiz, E.; Masegosa, R. M.; Hernandez-Fuentes, I. *Macromolecules* **1979**, *12*, 865.
(26) Suter, U. W. *J. Am. Chem. Soc.* **1979**, *101*, 6481.
(27) Sundararajan, P. R. *Macromolecules* **1980**, *13*, 512.
(28) Tanabe, T.; Koinuma, H.; Hirai, H. *Makromol. Chem.* **1981**, *182*, 3237.
(29) Abe, A. *Polym. J.* **1982**, *14*, 427.
(30) Mansfield, M. L., unpublished results.
(31) Reference 1, pp 108–110.
(32) Goldstein, H. "Classical Mechanics"; Addison-Wesley: Cambridge, MA, 1950; pp 118–124.
(33) Bur, A. J.; Fetters, L. J. *Chem. Rev.* **1976**, *76*, 727.
(34) Tonelli, A. E. *Macromolecules* **1974**, *7*, 628.
(35) Matsuo, K. *Macromolecules* **1977**, *10*, 498.
(36) Shieh, C.-F.; McNally, D.; Boyd, R. H. *Tetrahedron* **1969**, *25*, 3653.
(37) Reference 1, pp 64–65.
(38) Reference 1, pp 159–165.
(39) Bates, T. W.; Stockmayer, W. H. *Macromolecules* **1968**, *1*, 12.
(40) Bates, T. W.; Stockmayer, W. H. *Macromolecules* **1968**, *1*, 17.
(41) *J. Mol. Biol.* **1970**, *52*, 1.
(42) Schimmel, P. R.; Flory, P. J. *Proc. Natl. Acad. Sci. U.S.A.* **1967**, *58*, 52.
(43) Clark, D. S.; Dechter, J. J.; Mandelkern, L. *Macromolecules* **1979**, *12*, 626.
(44) Wu, C. C.; Komoroski, R. A.; Mandelkern, L. *Macromolecules* **1975**, *8*, 635.
(45) Lin, L.-N.; Brandts, J. F. *Biochemistry* **1980**, *19*, 3055.
(46) Levitt, M.; Lifson, S. *J. Mol. Biol.* **1969**, *46*, 269.
(47) Ambler, M. R.; McIntyre, D.; Fetters, L. J. *Macromolecules* **1978**, *11*, 300.
(48) Murakami, H.; Norisuye, T.; Fujita, H. *Macromolecules* **1980**, *13*, 345.
(49) Mansfield, M. L. *Macromolecules* **1981**, *14*, 1822.
(50) Troxell, T. C.; Scheraga, H. A. *Macromolecules* **1971**, *4*, 528.
(51) Han, C. C.; Yu, H. *Polym. Prepr., Am. Chem. Soc., Div. Polym. Chem.* **1973**, *14*, 121.
(52) Hays, J. B.; Magar, M. E.; Zimm, B. H. *Biopolymers* **1969**, *8*, 531.
(53) Schmid, C. W.; Rinehard, F. P.; Hearst, J. E. *Biopolymers* **1971**, *10*, 883.
(54) Landau, L. D.; Lifshitz, E. M. "Statistical Physics", 3rd ed., Pergamon Press: New York, 1980; pp 432–435.
(55) Mermin, N. D.; Wagner, H. *Phys. Rev. Lett.* **1966**, *17*, 1133.
(56) Hohenberg, P. C. *Phys. Rev.* **1967**, *158*, 383.
(57) Mermin, N. D. *Phys. Rev.* **1968**, *176*, 250.